

Potential estimates for quasi-linear parabolic equations

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Abstract

For a class of divergence type quasi-linear degenerate parabolic equations with a Radon measure on the right hand side we derive pointwise estimates for solutions via nonlinear Wolff potentials.

1 Introduction and main results

In this note we give a parabolic extension of a by now classical result by Kilpeläinen-Malý estimates [8] who proved pointwise estimates of solutions to quasi-linear p -Laplace type elliptic equations with measure in the right hand side. The estimates are expressed in terms of the nonlinear Wolff potential of the right hand side. These estimates were subsequently extended to fully nonlinear and subelliptic quasi-linear equation by Trudinger and Wang [13]. For the parabolic equations the corresponding result was recently given in [5, 6], but only for the "linear" case $p = 2$. Here we provide the estimates for parabolic equations in the degenerate case $p > 2$.

Let Ω be a domain in \mathbb{R}^n , $T > 0$. Let μ be a Radon measure on Ω . We are concerned with pointwise estimates for a class of non-homogeneous divergence type quasi-linear parabolic equations of the type

$$(1.1) \quad u_t - \operatorname{div} \mathbf{A}(x, t, u, \nabla u) = \mu \quad \text{in } \Omega_T = \Omega \times (0, T), \quad \Omega \subset \mathbb{R}^n,$$

and assume that the following structure conditions are satisfied:

$$(1.2) \quad \mathbf{A}(x, t, u, \zeta) \zeta \geq c_1 |\zeta|^p, \quad \zeta \in \mathbb{R}^n,$$

$$(1.3) \quad |\mathbf{A}(x, t, u, \zeta)| \leq c_2 |\zeta|^{p-1},$$

with some positive constants c_1, c_2 , whose model involves the parabolic p -Laplace equation

$$(1.4) \quad u_t - \Delta_p u = \mu, \quad (x, t) \in \Omega_T.$$

Before formulating the main results, let us remind the reader of the definition of a weak solution to equation (1.4).

We say that u is a weak solution to (1.4) if $u \in V(\Omega_T) := C([0, T]; L_{loc}^2(\Omega)) \cap L_{loc}^p(0, T; W_{loc}^{1,p}(\Omega))$ and for any compact subset \mathcal{K} of Ω and any interval $[t_1, t_2] \subset (0, T)$ the integral identity

$$(1.5) \quad \int_{\mathcal{K}} u \varphi dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\mathcal{K}} \{-u \varphi_\tau + \mathbf{A}(x, t, u, \nabla u) \nabla \varphi\} dx d\tau = \int_{t_1}^{t_2} \int_{\mathcal{K}} \varphi \mu(dx) d\tau.$$

for any $\varphi \in W_{loc}^{1,2}(0, T; L^2(\mathcal{K})) \cap L_{loc}^p(0, T; \dot{W}^{1,p}(\mathcal{K}))$.

Further on, we assume that $u_t \in L_{loc}^2(\Omega_T)$, since otherwise we can pass to Steklov averages.

The crucial role in our results is played by the truncated version of the Wolff potential defined by

$$(1.6) \quad W_p^\mu(x, R) = \int_0^R \left(\frac{\mu(B_r(x))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

In the sequel, γ stands for a constant which depends only on n, p, c_1, c_2 which may vary from line to line.

The main result of this paper is the following theorem.

Theorem 1.1. *Let u be a weak solution to equation (1.1). For every $\lambda \in (0, \frac{1}{n}]$ there exists $\gamma > 0$ depending on n, c_1, c_2 and λ , such that for almost all $(y, s) \in \Omega_T$ and for $\rho \in (0, 1)$ such that $B_{2\rho}(y) \times (s - 4\rho^2, s + 4\rho^2) \subset \Omega_T$ one has*

$$u(y, s) \leq \gamma \left\{ \left(\frac{1}{\rho^{p+n}} \iint_{B_\rho \times (s-\rho^p, s+\rho^p)} u_+^{(1+\lambda)(p-1)} dx dt \right)^{\frac{1}{1+\lambda(p-1)}} + 1 + W_p^\mu(y, 2\rho) \right\}.$$

The estimate above is not homogeneous in u which is usual for such type of equations [2, 4]. The proof of Theorem 1.1 is based on a suitable modifications of De Giorgi's iteration technique [1] following the adaptation of Kilpeläinen-Malý technique [8] to parabolic equations with ideas from [10, 12].

The rest of the paper contains the proof of the theorem.

2 Proof of Theorem 1.1

We start with some auxiliary integral estimates for the solutions of (1.1) which are formulated in the next lemma.

Define

$$G(u) = \begin{cases} u & \text{for } u > 1, \\ u^{2-2\lambda} & \text{for } 0 < u \leq 1. \end{cases}$$

Set

$$Q_\rho^{(\delta)}(y, s) = B_\rho(y) \times (s - \delta^{2-p}\rho^p, s + \delta^{2-p}\rho^p) \subset \Omega_T, \quad \rho \leq R.$$

Lemma 2.1. *Let the conditions of Theorem 1.1 be fulfilled. Let u be a solution to (1.4). Then there exists a constant $\gamma > 0$ depending only on n, p, c_1, c_2 such that for any $\varepsilon \in (0, 1)$, $l, \delta > 0$, any cylinder $Q_\rho^{(\delta)}(y, s)$ and any $\xi \in C_0^\infty(Q_\rho^{(\delta)}(y, s))$ such that $\xi(x, t) = 1$ for $(x, t) \in Q_{\rho/2}^{(\delta)}(y, s)$*

$$(2.1) \quad \begin{aligned} & \delta^2 \int_{L(t)} G\left(\frac{u(x, t) - l}{\delta}\right) \xi(x, t)^k dx + \iint_L \left(1 + \frac{u-l}{\delta}\right)^{-1+\lambda} \left(\frac{u-l}{\delta}\right)^{-2\lambda} |\nabla u|^p \xi(x, \tau)^k dx d\tau \\ & \leq \gamma \delta^2 \iint_L \left(\frac{u-l}{\delta}\right) |\xi_t| \xi^{k-1} dx d\tau + \gamma \frac{\delta^p}{\rho^p} \iint_L \left[\left(1 + \frac{u-l}{\delta}\right)^{1-\lambda} \left(\frac{u-l}{\delta}\right)^{2\lambda} \right]^{p-1} \xi^{k-p} dx d\tau \\ & \quad + \gamma \delta^{3-p} \mu(B_\rho(y)), \end{aligned}$$

where $L = Q_\rho^{(\delta)}(y, s) \cap \{u > l\}$, $L(t) = L \cap \{\tau = t\}$ and $\lambda \in (0, 1)$, $k > p$.

Proof. First, note that

$$(2.2) \quad \int_l^u \left(1 + \frac{s-l}{\delta}\right)^{-1+\lambda} \left(\frac{s-l}{\delta}\right)^{-2\lambda} ds \leq \gamma \delta,$$

and

$$\begin{aligned}
& \int_l^u dw \int_l^w \left(1 + \frac{s-l}{\delta}\right)^{-1+\lambda} \left(\frac{s-l}{\delta}\right)^{-2\lambda} ds = \int_l^u \left(1 + \frac{s-l}{\delta}\right)^{-1+\lambda} \left(\frac{s-l}{\delta}\right)^{-2\lambda} (u-s) ds \\
& \geq \frac{1}{2}(u-l) \int_l^{\frac{u+l}{2}} \left(1 + \frac{s-l}{\delta}\right)^{-1+\lambda} \left(\frac{s-l}{\delta}\right)^{-2\lambda} ds = \frac{\delta^2}{2} \left(\frac{u-l}{\delta}\right) \int_0^{\frac{u-l}{2\delta}} (1+z)^{-1+\lambda} z^{-2\lambda} dz \\
(2.3) \quad & \geq \gamma \delta^2 G\left(\frac{u-l}{\delta}\right).
\end{aligned}$$

Test (1.5) by φ defined by

$$(2.4) \quad \varphi(x, t) = \left[\int_l^{u(x,t)} \left(1 + \frac{s-l}{\delta}\right)^{-1+\lambda} \left(\frac{s-l}{\delta}\right)^{-2\lambda} ds \right]_+ \xi(x, t)^k,$$

and $t_1 = s - \delta^{2-p} \rho^p$, $t_2 = t$. Using the Young inequality and (2.2) we have for any $t > 0$

$$\begin{aligned}
& \int_{L(t)} \int_l^u dw \int_l^w \left(1 + \frac{s-l}{\delta}\right)^{-1+\lambda} \left(\frac{s-l}{\delta}\right)^{-2\lambda} ds \xi^k dx \\
& + \iint_L \left(1 + \frac{u-l}{\delta}\right)^{-1+\lambda} \left(\frac{u-l}{\delta}\right)^{-2\lambda} |\nabla u|^p \xi^k dx dt \\
& \leq \gamma \iint_L \int_l^u dw \int_l^w \left(1 + \frac{s-l}{\delta}\right)^{-1+\lambda} \left(\frac{s-l}{\delta}\right)^{-2\lambda} ds |\xi_t| \xi^{k-1} dx dt \\
& + \gamma \iint_L \left[\left(1 + \frac{u-l}{\delta}\right)^{1-\lambda} \left(\frac{u-l}{\delta}\right)^{2\lambda} \right]^{p-1} \xi^{k-p} dx dt + \gamma \delta^{3-p} \rho^p \mu(B_\rho(y)).
\end{aligned}$$

From this using (2.2) and (2.3) we obtain the required (2.1). \square

Now set

$$(2.5) \quad \psi(x, t) = \frac{1}{\delta} \left[\int_l^{u(x,t)} \left(1 + \frac{s-l}{\delta}\right)^{-\frac{1-\lambda}{p}} \left(\frac{s-l}{\delta}\right)^{-\frac{2\lambda}{p}} ds \right]_+.$$

The next lemma is a direct consequence of Lemma 2.1.

Lemma 2.2. *Let the conditions of Lemma 2.1 be fulfilled. Then*

$$\begin{aligned}
& \int_{L(t)} G\left(\frac{u-l}{\delta}\right) \xi^k dx + \delta^{p-2} \iint_L |\nabla \psi|^p \xi^k dx d\tau \\
(2.6) \quad & \leq \gamma \frac{\delta^{p-2}}{\rho^p} \iint_L \left(1 + \frac{u-l}{\delta}\right)^{(1-\lambda)(p-1)} \left(\frac{u-l}{\delta}\right)^{2\lambda(p-1)} \xi^{k-p} dx d\tau + \gamma \frac{\rho^p}{\delta^{p-1}} \mu(B_\rho(y)).
\end{aligned}$$

Let (y, s) be an arbitrary point in Ω_T . Let $R \leq \frac{1}{2} \min \left\{ 1, \text{dist}(y, \partial\Omega), s^{\frac{1}{2}}, (T-s)^{\frac{1}{2}} \right\}$ and $Q_R(y, s) = B_R(y) \times (s - R^2, s + R^2)$. Fix $\rho \leq R$ and for $j = 0, 1, 2, \dots$ set

$$\rho_j = \rho 2^{-j}, \quad Q_j = B_j \times (s - \delta_j^{2-p} \rho_j^p, s + \delta_j^{2-p} \rho_j^p), \quad B_j = B_{\rho_j}(y), \quad L_j = Q_j \cap \Omega_T \cap \{u(x, t) > l_j\}.$$

Let $\xi_j \in C_0^\infty(Q_j)$ be such that $\xi_j(x, t) = 1$ for $(x, t) \in B_{j+1} \times (s - \frac{3}{4} \delta_j^{2-p} \rho_j^p, s + \frac{3}{4} \delta_j^{2-p} \rho_j^p)$, $|\nabla \xi_j| \leq \gamma \rho_j^{-1}$, $|\frac{\partial \xi_j}{\partial t}| \leq \gamma \delta_j^{p-2} \rho_j^{-p}$.

The sequences of positive numbers $(l_j)_{j \in \mathbb{N}}$ and $(\delta_j)_{j \in \mathbb{N}}$ are defined inductively as follows.

Set $l_0 = 0$ and assume that l_1, l_2, \dots, l_j and $\delta_0, \delta_1, \dots, \delta_{j-1}$ have been already chosen in such a way that $\delta_k = l_{k+1} - l_k$. Let us show how to chose l_{j+1} and δ_j .

For $l \geq l_j + \rho_j$ set

$$(2.7) \quad \begin{aligned} A_j(l) &= \frac{(l - l_j)^{p-2}}{\rho_j^{n+p}} \iint_{\tilde{L}_j} \left(\frac{u - l_j}{l - l_j} \right)^{(1+\lambda)(p-1)} \xi_j^{k-p} dx d\tau \\ &+ \sup_{|t-s| \leq (l-l_j)^{2-p} \rho_j^p} \frac{1}{\rho_j^n} \int_{\tilde{L}_j(t)} G \left(\frac{u - l_j}{l - l_j} \right) \xi_j^k dx, \end{aligned}$$

where $\tilde{L}_j = \tilde{Q}_j \cap \Omega_T \cap \{u(x, t) > l_j\}$, $\tilde{Q}_j = B_j \times (s - (l - l_j)^{2-p} \rho_j^p, s + (l - l_j)^{2-p} \rho_j^p)$.

Fix a number $\varkappa \in (0, 1)$ depending on n, p, c_1, c_2 , which will be specified later. Set $\hat{\delta}_0 = \max\{1, \rho_0\}$, $\hat{\delta}_j = \rho_j$. For $j = 0, 1, 2, \dots$, if

$$(2.8) \quad A_j(l_j + \hat{\delta}_j) \leq \varkappa,$$

we set $\delta_j = \hat{\delta}_j$ and $l_{j+1} = l_j + \delta_j$.

Note that $A_j(l)$ is continuous as a function of l and $A_j(l) \searrow 0$ as $l \rightarrow \infty$. So if

$$(2.9) \quad A_j(l_j + \hat{\delta}_j) > \varkappa,$$

there exists $\bar{l} > l_j + \hat{\delta}_j$ such that $A_j(\bar{l}) = \varkappa$. In this case we set $l_{j+1} = \bar{l}$ and $\delta_j = l_{j+1} - l_j$.

Note that our choices guarantee that $\tilde{Q}_j \subset Q_R(y, s)$ and

$$(2.10) \quad A_j(l_{j+1}) \leq \varkappa.$$

The following lemma is a key in the Kilpeläinen-Malý technique [8].

Lemma 2.3. *Let the conditions of Theorem 1.1 be fulfilled. There exists $\gamma > 0$ depending on the data, such that for all $j \geq 1$ we have*

$$(2.11) \quad \delta_j \leq \frac{1}{2} \delta_{j-1} + \rho_j + \gamma \left(\frac{1}{\rho_j^{n-p}} \mu(B_j) \right)^{\frac{1}{p-1}}.$$

Proof. Fix $j \geq 1$. Without loss assume that

$$(2.12) \quad \delta_j > \frac{1}{2} \delta_{j-1}, \quad \delta_j > \rho_j,$$

since otherwise (2.11) is evident. The second inequality in (2.12) guarantees that $A_j(l_{j+1}) = \varkappa$ and $\tilde{Q}_j = Q_j$.

Next we claim that under conditions (2.12) there is a $\gamma > 0$ such that

$$(2.13) \quad \delta_j^{p-2} \rho_j^{-(p+n)} |L_j| \leq \gamma \varkappa.$$

Indeed, for $(x, t) \in L_j$ one has

$$(2.14) \quad \frac{u(x, t) - l_{j-1}}{\delta_{j-1}} = 1 + \frac{u(x, t) - l_j}{\delta_{j-1}} \geq 1.$$

Note that the first inequality in (2.12) yields $\xi_{j-1} = 1$ on Q_j . Hence

$$\begin{aligned} \delta_j^{p-2} \rho_j^{-(p+n)} |L_j| &\leq \delta_j^{p-2} \rho_j^{-(p+n)} \iint_{L_j} G \left(\frac{u - l_{j-1}}{\delta_{j-1}} \right) \xi_{j-1}^k dx d\tau \\ &\leq \rho_j^{-n} \sup_{|t-s| \leq \delta_j^{2-p} \rho_j^p} \int_{L_j(t)} G \left(\frac{u - l_{j-1}}{\delta_{j-1}} \right) \xi_{j-1}^k dx \leq 2^n \rho_{j-1}^{-n} \sup_{|t-s| \leq \delta_{j-1}^{2-p} \rho_{j-1}^p} \int_{L_{j-1}(t)} G \left(\frac{u - l_{j-1}}{\delta_{j-1}} \right) \xi_{j-1}^k dx \leq 2^n \varkappa, \end{aligned}$$

which proves the claim.

Let us estimate the terms in the right hand side of (2.7) with $l = l_{j+1}$. For this we decompose L_j as $L_j = L'_j \cup L''_j$,

$$(2.15) \quad L'_j = \left\{ (x, t) \in L_j : \frac{u(x, t) - l_j}{\delta_j} < \varepsilon \right\}, \quad L''_j = L_j \setminus L'_j,$$

where $\varepsilon \in (0, 1)$ depending on n, p, c_1, c_2 is small enough to be determined later.

By (2.13) we have

$$(2.16) \quad \frac{\delta_j^{p-2}}{\rho_j^{n+p}} \iint_{L'_j} \left(\frac{u - l_j}{\delta_j} \right)^{(1+\lambda)(p-1)} \xi_j^{k-p} dx d\tau \leq \frac{\delta_j^{p-2}}{\rho_j^{n+p}} \iint_{L'_j} \varepsilon^{(1+\lambda)(p-1)} dx d\tau \leq 2^n \varepsilon^{(1+\lambda)(p-1)} \varkappa.$$

Set

$$(2.17) \quad \psi_j(x, t) = \frac{1}{\delta_j} \left(\int_{l_j}^{u(x, t)} \left(1 + \frac{s - l_j}{\delta_j} \right)^{-\frac{1-\lambda}{p}} \left(\frac{s - l_j}{\delta_j} \right)^{-\frac{2\lambda}{p}} ds \right)_+,$$

and

$$\rho(\lambda) = \frac{p}{p-1-\lambda}.$$

Note that $\lambda \leq \frac{1}{n}$ due to the assumption.

The following inequalities are easy to verify

$$(2.18) \quad c\psi_j(x, t)^{\rho(\lambda)} \leq \left(\frac{u(x, t) - l_j}{\delta_j} \right) \text{ for } (x, t) \in L_j, \text{ and}$$

$$(2.19) \quad \left(\frac{u(x, t) - l_j}{\delta_j} \right) \leq c(\varepsilon)\psi_j(x, t)^{\rho(\lambda)}, \quad (x, t) \in L''_j.$$

Hence

$$(2.20) \quad \begin{aligned} & \frac{\delta_j^{p-2}}{\rho_j^{n+p}} \iint_{L''_j} \left(\frac{u - l_j}{\delta_j} \right)^{(1+\lambda)(p-1)} \xi_j^{k-p} dx d\tau \\ & \leq \gamma(\varepsilon) \frac{\delta_j^{p-2}}{\rho_j^{n+p}} \iint_{L''_j} \psi_j^{\frac{n+\rho(\lambda)}{n}} \xi_j^{k-p} dx d\tau. \end{aligned}$$

The integral in the second terms of the right hand side of (2.20) is estimated by using the Gagliardo–Nirenberg inequality in the form [9, Chapter II, Theorem 2.1] as follows

$$(2.21) \quad \begin{aligned} & \gamma(\varepsilon) \frac{\delta_j^{p-2}}{\rho_j^{n+p}} \iint_{L''_j} \psi_j^{\frac{n+\rho(\lambda)}{n}} \xi_j^{k-p} dx d\tau \\ & \leq \gamma \left(\sup_{|t-s| \leq \delta_j^{2-p} \rho_j^p} \frac{1}{\rho_j^n} \int_{L_j(t)} \psi_j^{\rho(\lambda)} dx \right)^{\frac{p}{n}} \left(\frac{1}{\rho_j^n} \iint_{L_j} \left| \nabla \left(\psi_j \xi_j^{\frac{(k-p)n}{p(n+\rho(\lambda))}} \right) \right|^p dx d\tau \right). \end{aligned}$$

Let us estimate separately the first factor in the right hand side of (2.21).

$$(2.22) \quad \begin{aligned} & \sup_{|t-s| \leq \delta_j^{2-p} \rho_j^p} \int_{L_j(t)} \psi_j^{\rho(\lambda)} dx \stackrel{\text{by (2.18)}}{\leq} c^{-1} \sup_{|t-s| \leq \delta_j^{2-p} \rho_j^p} \int_{L_j(t)} \frac{u - l_j}{\delta_j} dx \\ & \stackrel{\text{by (2.12)}}{\leq} 2c^{-1} \sup_{|t-s| \leq \delta_j^{2-p} \rho_j^p} \int_{L_j(t)} \frac{u - l_{j-1}}{\delta_{j-1}} \xi_{j-1}^k dx \\ & \stackrel{\text{by (2.14)}}{\leq} 2c^{-1} \sup_{|t-s| \leq \delta_{j-1}^{2-p} \rho_{j-1}^p} \int_{L_{j-1}(t)} G \left(\frac{u - l_{j-1}}{\delta_{j-1}} \right) \xi_{j-1}^k dx \\ & \stackrel{\text{by (2.10)}}{\leq} 2c^{-1} \varkappa \rho_{j-1}^n = \gamma \rho_j^n \varkappa. \end{aligned}$$

Combining (2.20), (2.21) and (2.22) we obtain

$$\begin{aligned}
& \frac{\delta_j^{p-2}}{\rho_j^{n+p}} \iint_{L_j''} \left(\frac{u-l_j}{\delta_j} \right)^{(1+\lambda)(p-1)} \xi_j^{k-p} dx d\tau \\
(2.23) \quad & \leq \gamma(\varepsilon) \varkappa^{\frac{p}{n}} \delta_j^{p-2} \rho_j^{-n} \iint_{L_j} \left| \nabla \left(\psi_j \xi_j^{\frac{(k-p)n}{p(n+\rho(\lambda))}} \right) \right|^p dx d\tau.
\end{aligned}$$

For the last term in the above inequality we estimate by (2.13) and (2.18)

$$\begin{aligned}
& \delta_j^{p-2} \rho_j^{-n} \iint_{L_j} \psi_j^p |\nabla \xi_j|^p dx d\tau \leq \gamma \delta_j^{p-2} \rho_j^{-n-p} \iint_{L_j} \psi_j^p dx d\tau \\
& \leq \gamma \delta_j^{p-2} \rho_j^{-n-p} \iint_{L_j} \left(\frac{u-l_j}{\delta_j} \right)^{p-1-\lambda} dx d\tau \stackrel{\text{by (2.12)}}{\leq} \gamma \delta_{j-1}^{p-2} \rho_j^{-n-p} \iint_{L_j} \left(\frac{u-l_{j-1}}{\delta_{j-1}} \right)^{p-1-\lambda} \xi_{j-1}^{k-p} dx d\tau \\
(2.24) \quad & \leq \gamma \delta_{j-1}^{p-2} \rho_j^{-n-p} \iint_{L_{j-1}} \left(\frac{u-l_{j-1}}{\delta_{j-1}} \right)^{(p-1)(1+\lambda)} \xi_{j-1}^{k-p} dx d\tau \leq \gamma \varkappa.
\end{aligned}$$

By Lemma 2.2

$$\begin{aligned}
& \frac{1}{\rho_j^n} \int_{L_j(t)} G \left(\frac{u-l_j}{\delta_j} \right) \xi_j^k dx + \frac{\delta_j^{p-2}}{\rho_j^n} \iint_{L_j} |\nabla \psi_j|^p \xi_j^k dx d\tau \\
& \leq \gamma \frac{\delta_j^{p-2}}{\rho_j^{p+n}} \iint_{L_j} \left(1 + \frac{u-l_j}{\delta_j} \right)^{(1-\lambda)(p-1)} \left(\frac{u-l_j}{\delta_j} \right)^{2\lambda(p-1)} \xi_j^{k-p} dx d\tau \\
(2.25) \quad & + \gamma \frac{\rho_j^{p-n}}{\delta_j^{p-1}} \mu(B_{\rho_j}(y)).
\end{aligned}$$

Using the decomposition (2.15) and the first inequality in (2.12) we have

$$\begin{aligned}
& \delta_j^{p-2} \rho_j^{-(n+p)} \iint_{L_j} \left(1 + \frac{u-l_j}{\delta_j} \right)^{(1-\lambda)(p-1)} \left(\frac{u-l_j}{\delta_j} \right)^{2\lambda(p-1)} dx d\tau \\
& \leq \gamma \varepsilon^{2\lambda(p-1)} \delta_j^{p-2} \rho_j^{-(n+p)} |L_j| + \gamma(\varepsilon) \delta_{j-1}^{p-2} \rho_{j-1}^{-(n+p)} \iint_{L_{j-1}} \left(\frac{u-l_{j-1}}{\delta_{j-1}} \right)^{(1+\lambda)(p-1)} dx d\tau \\
(2.26) \quad & \leq \gamma \varepsilon^{2\lambda(p-1)} \varkappa + \gamma(\varepsilon) \varkappa.
\end{aligned}$$

Thus we obtain the following estimate for the first term of $A_j(l_{j+1})$:

$$\begin{aligned}
& \frac{\delta_j^{p-2}}{\rho_j^{n+p}} \iint_{L_j} \left(\frac{u-l_j}{\delta_j} \right)^{(1+\lambda)(p-1)} dx d\tau \\
(2.27) \quad & \leq \gamma \varepsilon^{2\lambda(p-1)} \varkappa + \gamma(\varepsilon) \varkappa^{\frac{p}{n}} \left(\varkappa + \delta_j^{1-p} \rho_j^{p-n} \mu(B_j) \right).
\end{aligned}$$

Let us estimate the second term in the right hand side of (2.7). By (2.25) we have

$$\begin{aligned}
& \sup_{|t-s| \leq \delta_j^{2-p} \rho_j^p} \rho_j^{-n} \int_{L_j(t)} G \left(\frac{u-l_j}{\delta_j} \right) \xi_j^k dx \\
& \leq \delta_j^{p-2} \rho_j^{-(n+p)} \iint_{L_j} \left(1 + \frac{u-l_j}{\delta_j} \right)^{(1-\lambda)(p-1)} \left(\frac{u-l_j}{\delta_j} \right)^{2\lambda(p-1)} \xi_j^{k-p} dx d\tau + \gamma \delta_j^{1-p} \rho_j^{p-n} \mu(B_j) \\
& \quad (\text{by using the decomposition (2.15) and (2.27)}) \\
(2.28) \quad & \leq \gamma \varepsilon^{2\lambda(p-1)} \varkappa + \gamma(\varepsilon) \varkappa^{\frac{p}{n}} \left(\varkappa + \delta_j^{1-p} \rho_j^{p-n} \mu(B_j) \right) + \gamma \delta_j^{1-p} \rho_j^{p-n} \mu(B_j).
\end{aligned}$$

Combining (2.26) and (2.28) and choosing ε appropriately we can find γ_1 and γ such that

$$(2.29) \quad \varkappa \leq \gamma_1 \varkappa^{\frac{p}{n}} \left(\varkappa + \delta_j^{1-p} \rho_j^{p-n} \mu(B_j) \right) + \gamma \delta_j^{1-p} \rho_j^{p-n} \mu(B_j).$$

Now choosing $\varkappa < 1$ such that $\varkappa^{\frac{p}{n}} = \frac{1}{2\gamma_1}$ we have

$$(2.30) \quad \delta_j \leq \gamma (\rho_j^{p-n} \mu(B_j))^{\frac{1}{p-1}},$$

which completes the proof of the lemma. \square

In order to complete the proof of Theorem 1.1 we sum up (2.11) with respect to j from 1 to $J-1$

$$(2.31) \quad \begin{aligned} l_J &\leq \gamma \delta_0 + \gamma \sum_{j=1}^{\infty} \rho_j + \gamma \sum_{j=1}^{\infty} (\rho_j^{p-n} \mu(B_j))^{\frac{1}{p-1}} \\ &\leq \gamma (\delta_0 + \rho + W_p^\mu(y, 2\rho)). \end{aligned}$$

Let us estimate δ_0 . There are two cases to consider. If $l_1 = \hat{\delta}_0 = \max\{1, \rho\}$ then $\delta_0 = \max\{1, \rho\}$. If on the other hand l_1 and δ_0 are defined by $A_0(l_1) = \varkappa$ then by (2.7)

$$\varkappa = \frac{\delta_0^{p-2}}{\rho^{n+p}} \iint_{Q_\rho^{(\delta_0)}} \left(\frac{u_+}{\delta_0} \right)^{(1+\lambda)(p-1)} \xi_0^{k-p} dx d\tau + \sup_{|t-s| < \delta_0^{2-p} \rho^p} \rho^{-n} \int_{B_\rho} G \left(\frac{u_+}{\delta_0} \right) \xi_0^k dx.$$

Using the decomposition (2.15) with ε chosen via \varkappa , and Lemma 2.2 one can see that

$$\sup_{|t-s| < \delta_0^{2-p} \rho^p} \rho^{-n} \int_{B_\rho} G \left(\frac{u_+}{\delta_0} \right) dx \leq \varkappa/2 + \frac{\delta_0^{p-2}}{\rho^{n+p}} \iint_{Q_\rho^{(\delta_0)}} \left(\frac{u_+}{\delta_0} \right)^{(1+\lambda)(p-1)} dx d\tau.$$

Note that $\delta_0 \geq \max\{1, \rho\}$, thus $\delta_0^{2-p} \rho^p \leq \rho^p$. Hence we obtain

$$\varkappa \leq \gamma \frac{\delta_0^{p-2}}{\rho^{n+p}} \iint_{Q_\rho^{(\rho)}} \left(\frac{u_+}{\delta_0} \right)^{(1+\lambda)(p-1)} dx d\tau.$$

Combining this with the first case we have

$$(2.32) \quad \delta_0 \leq \gamma \left\{ \left(\frac{1}{\rho^{p+n}} \iint_{B_\rho(y) \times (s-\rho^p, s+\rho^p)} u_+^{(1+\lambda)(p-1)} dx dt \right)^{\frac{1}{1+\lambda(p-1)}} + 1 + \rho \right\}.$$

Hence the sequence $(l_j)_{j \in \mathbb{N}}$ is convergent, and $\delta_j \rightarrow 0$ ($j \rightarrow \infty$), and we can pass to the limit $J \rightarrow \infty$ in (2.31). Let $l = \lim_{j \rightarrow \infty} l_j$. From (2.10) we conclude that

$$(2.33) \quad \frac{1}{\rho_j^{n+p}} \iint_{Q_j} (u - l)_+^{(1+\lambda)(p-1)} \leq \gamma \varkappa \delta_j^{1+\lambda(p-1)} \rightarrow 0 \quad (j \rightarrow \infty).$$

Choosing (y, s) as a Lebesgue point of the function $(u - l)_+^{(1+\lambda)(p-1)}$ we conclude that $u(y, s) = l$ and hence $u(y, s)$ is estimated from above by

$$u(y, s) \leq \gamma \left\{ \left(\frac{1}{\rho^{p+n}} \iint_{B_\rho(y) \times (s-\rho^p, s+\rho^p)} u_+^{(1+\lambda)(p-1)} dx dt \right)^{\frac{1}{1+\lambda(p-1)}} + 1 + \rho + W_p^\mu(y, 2\rho) \right\}$$

Applicability of the Lebesgue differentiation theorem follows from [7, Chap. II, Sec. 3].

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